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Note on Polynomial Approximation of Monomials and Diophantine Approximation

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1. THEOREM

This note establishes a connection between the rate of approximation of certain monomials and diophantine approximation. Let $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$. Let $d \in \mathbb{R}$. For $1 \leqslant p \leqslant \infty$ and large positive integers n, let

$$e_{n,p}(\alpha, d) = \min_{\deg(P) \leqslant n} \left(\int_0^1 |x^{n\alpha + d} - P(x)|^p \, dx \right)^{1/p}.$$

Thus $e_{n,p}(\alpha, d)$ is the error in best approximation in the L_p norm of $x^{n\alpha + d}$ by polynomials of degree at most n. Let

$$H(\alpha) = \exp\left(\int_0^1 \log \left| \frac{\alpha - x}{\alpha + x} \right| dx \right).$$

In [5, Lemma 5.4], it was shown that

$$\lim_{n \to \infty} \{e_{n,2}(\alpha, d)\}^{1/n} = H(\alpha) \qquad \text{if} \quad \operatorname{Re}(\alpha) > 0 \text{ and } \alpha \notin (0, 1].$$

The purpose of this note is to clarify what happens if $\alpha \in (0, 1)$. It turns out that (at least for the case where d is an integer), the behaviour of $e_{n,p}(\alpha, d)$ as $n \to \infty$ depends on how well α can be approximated by rationals. Suppose d is an integer. If α is rational, we see $e_{n,p}(\alpha, d) = 0$ for infinitely many n, while if α is irrational, we see $e_{n,p}(\alpha, d) > 0$ for all n. If $\alpha = 1$, we see $e_{n,p}(\alpha, d) = 0$ for all large enough n, provided d is non-positive.

For each $\rho \in (0, 1)$ and real d, we let $E(\rho, d) = \{x \in (0, 1)$: for infinitely

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many n, there exists $j \leqslant n$ satisfying $|x-(j-d)/n| < \rho^n\}$ and $E_d = \bigcup_{\rho \in (0,1)} E(\rho, d)$.

Note that if $0 < \rho < \rho' < 1$, then $E(\rho, d) \subset E(\rho', d)$. Further if $[d]$ is the integer part and d' the fractional part of d, then every approximation $(j-d)/n$ to x yields an approximation $((j-[d])-d')/n$ to x and conversely. Consequently $E_d = E_{d'}$. In particular, when d is an integer, $E_d = E_0$. The latter is the set of numbers in (0, 1) which can be approximated by rationals faster than a geometric sequence. Because of their exceptionally strong approximation properties by rationals, the irrational elements of E_0 are all transcendental (see [2, pp. 158–161]).

THEOREM. *Let* $\alpha \in (0, 1)$, $d \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$.

(i) $\limsup_{n \to \infty} \{e_{n,p}(\alpha, d)\}^{1/n} = H(\alpha)$;

(ii) $\liminf_{n \to \infty} \{e_{n,p}(\alpha, d)\}^{1/n} = H(\alpha)\, \mu(\alpha, d)$,

where

$$\begin{aligned} \mu(\alpha, d) &= \inf\{\rho : \alpha \in E(\rho, d)\} && \textit{if} \quad \alpha \in E_d \\ &= 1 && \textit{if} \quad \alpha \notin E_d; \end{aligned}$$

(iii) $\lim_{n \to \infty} \{e_{n,p}(\alpha, d)\}^{1/n} = H(\alpha)$ *iff* $\alpha \notin E_d$;

(iv) E_d *has logarithmic dimension* $\leqslant 2$ (*and hence Hausdorff dimension* 0). *If d is an integer, E_d has logarithmic dimension* 2.

2. PROOF OF THE THEOREM

We shall need some lemmas. Within the next three lemmas α and d are fixed. For each positive integer n, we let $l(n)$ be the (eventually positive) integer such that

$$\Delta_n = \left|\alpha - \frac{l(n)-d}{n}\right| = \min_{0 \leqslant j \leqslant n} \left|\alpha - \frac{j-d}{n}\right|.$$

LEMMA 1. *Let* $\mathscr{L} = \{n : n\alpha + d \textit{ is an integer}\}$. *Then*

$$\lim_{\substack{n \to \infty \\ n \notin \mathscr{L}}} \{e_{n,2}(\alpha, d)/\Delta_n\}^{1/n} = H(\alpha).$$

Proof. Let $n \notin \mathscr{S}$. By the Gram formula [1, p. 196],

$$e_{n,2}(\alpha, d) = (2n\alpha + 2d + 1)^{-1/2} \prod_{j=0}^{n} \left|\frac{n\alpha + d - j}{n\alpha + d + j + 1}\right|.$$

Let t be the smallest integer larger than $|d|$. We see

$$n^{-1}\log e_{n,2}(\alpha, d) = n^{-1}\sum_{j=t}^{n-t}\log\left|\frac{\alpha-(j-d)/n}{\alpha+(j+d)/n}\right| + o(1)$$
$$= \int_0^1 \log\left(\frac{1}{\alpha+x}\right)dx + n^{-1}\sum_{j=t}^{n-t}\log|\alpha-(j-d)/n| + o(1) \qquad (1)$$

as $\log(1/(\alpha+x))$ is continuous in $[0, 1]$ and by the theory of Riemann sums. Next if $\alpha \in ((k-1-d)/n, (k-d)/n)$, monotonicity of $\log|\alpha - x|$ in $[0, \alpha)$ and $(\alpha, 1]$ yields

$$\int_{(t-d)/n}^{(k-1-d)/n}\log|\alpha-x|\,dx \leqslant n^{-1}\sum_{j=t}^{k-2}\log\left|\alpha-\frac{(j-d)}{n}\right|$$
$$\leqslant \int_{(t-1-d)/n}^{(k-2-d)/n}\log|\alpha-x|\,dx$$
$$\int_{(k-d)/n}^{(n-t-d)/n}\log|\alpha-x|\,dx \leqslant n^{-1}\sum_{j=k+1}^{n-t}\log\left|\alpha-\frac{(j-d)}{n}\right|$$
$$\leqslant \int_{(k+1-d)/n}^{(n-t+1-d)/n}\log|\alpha-x|\,dx.$$

Hence, adding, we obtain

$$n^{-1}\sum_{\substack{j=t\\ j\neq k-1,k}}^{n-t}\log\left|\alpha-\frac{(j-d)}{n}\right| = \int_0^1\log|\alpha-x|\,dx + o(1). \qquad (2)$$

Here k depends on n, of course. Now one of $(k-d)/n$, $(k-1-d)/n$ is $(l(n)-d)/n$; the other is at least a distance of $1/(2n)$ from α, and so for either $j=k-1$ or $j=k$, we have

$$n^{-1}\log\left|\alpha-\frac{(j-d)}{n}\right| = O(n^{-1}\log n) = o(1).$$

Together with (1), (2) this yields for all $n \notin \mathscr{L}$

$$n^{-1}\log e_{n,2}(\alpha, d) = \int_0^1\log\left|\frac{\alpha-x}{\alpha+x}\right|dx + n^{-1}\log\left|\alpha-\frac{l(n)-d}{n}\right| + o(1). \quad \blacksquare$$

It is also possible to deduce Lemma 1 from known results on convergence of Riemann sums for singular integrands.

LEMMA 2. (i) $\liminf_{n\to\infty} \Delta_n^{1/n} = \mu(\alpha, d)$.

(ii) $\limsup_{n\to\infty} \Delta_n^{1/n} = 1$.

(iii) $\lim_{n\to\infty} \Delta_n^{1/n} = 1$ *iff* $\alpha \notin E_d$.

Proof. Part (i) is immediate from the definition of $\mu(\alpha, d)$, $E(\rho, d)$, and E_d. Part (ii) is immediate if $\alpha \notin E_d$. Suppose now $\alpha \in E_d$. For some infinite sequence of positive integers, and some $\rho \in (0, 1)$,

$$\Delta_n < \rho^n, \qquad n \in \mathbb{N}$$

$$\Rightarrow \Delta_{n+1} \geqslant \left|\frac{l(n+1)-d}{n+1} - \frac{l(n)-d}{n}\right| - \Delta_n$$

$$> |n\{l(n+1)-l(n)\} + d - l(n)|\, n^{-1}(n+1)^{-1} - \rho^n.$$

Now if $l(n) \geqslant l(n+1)$, then for large n,

$$n\{l(n+1)-l(n)\} + d - l(n) \leqslant d - l(n) < -1.$$

Further if $l(n) < l(n+1)$, then for large n,

$$n\{l(n+1)-l(n)\} + d - l(n) \geqslant n + d - l(n) > 1.$$

Thus

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \Delta_{n+1}^{1/(n+1)} \geqslant \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \{n^{-1}(n+1)^{-1} - \rho^n\}^{1/(n+1)} = 1.$$

Part (iii) follows from (i) and (ii). ∎

For $p = 2$, parts (i), (ii), and (iii) of the Theorem now follow. For general $1 \leqslant p \leqslant \infty$, one uses

LEMMA 3. (i) *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then*

$$e_{n,p}(\alpha, d) \leqslant e_{n,q}(\alpha, d).$$

(ii) $e_{n,\infty}(\alpha, d) \leqslant |n\alpha + d|\, e_{n-1,1}(\alpha, \alpha + d - 1)$.

Proof. Part (i) follows from monotonicity of the $L_p[0, 1]$ norm [6, pp. 16, 25, Theorems 10–12(i)], in p.

(ii) Let $Q(x)$ be a polynomial of degree $\leqslant n-1$, and let $P(x) = \int_0^x (n\alpha + d)\, Q(u)\, du$, so that $P(x)$ is of degree $\leqslant n$ and $P(0) = 0$. Let $x \in [0, 1]$. Then

$$x^{n\alpha+d} - P(x) = \int_0^x \{(n\alpha + d)\, u^{n\alpha+d-1} - P'(u)\}\, du$$

so

$$\max_{x\in[0,1]} |x^{n\alpha+d} - P(x)| \leqslant |n\alpha + d| \int_0^1 |u^{(n-1)\alpha+(\alpha+d-1)} - Q(u)|\, du.$$

Then taking the infimum over all Q, the result follows. ∎

For $1 \leqslant p \leqslant \infty$, part (i) of the Theorem now follows. We prove (ii), which is harder. If $2 \leqslant p \leqslant \infty$, Lemma 3 yields

$$\begin{aligned} H(\alpha)\,\mu(\alpha, d) &= \liminf_{n\to\infty} \{e_{n,2}(\alpha, d)\}^{1/n} \\ &\leqslant \liminf_{n\to\infty} \{e_{n,p}(\alpha, d)\}^{1/n} \\ &\leqslant \liminf_{n\to\infty} \{e_{n-1,2}(\alpha, \alpha+d-1)\}^{1/n} = H(\alpha)\,\mu(\alpha, \alpha+d-1). \end{aligned}$$

Similarly, if $1 \leqslant p \leqslant 2$, one obtains

$$H(\alpha)\,\mu(\alpha, d) \geqslant \liminf_{n\to\infty} \{e_{n,p}(\alpha, d)\}^{1/n} \geqslant H(\alpha)\,\mu(\alpha, d-\alpha+1).$$

We deduce

$$\mu(\alpha, d-\alpha+1) \leqslant \mu(\alpha, d) \leqslant \mu(\alpha, d+\alpha-1), \tag{3}$$

and note that (ii) of the Theorem follows if $\leqslant$ can be replaced by $=$ in (3). To this end, let $\rho > \mu(\alpha, d-\alpha+1)$. For infinitely many n, there exists $j \leqslant n$ such that

$$\begin{aligned} &\left|\alpha - \frac{j-(d-\alpha+1)}{n}\right| < \rho^n \\ &\Rightarrow \left|\alpha\left(1-\frac{1}{n}\right) - \frac{j-1-d}{n}\right| < \rho^n \\ &\Rightarrow \left|\alpha - \frac{j-1-d}{n-1}\right| < \frac{n}{n-1}\rho^n < \rho^{n-1}, \qquad n \text{ large enough.} \end{aligned}$$

We deduce $\alpha \in E(\rho, d)$ for any $\rho > \mu(\alpha, d-\alpha+1)$ so that $\mu(\alpha, d) \leqslant \mu(\alpha, d-\alpha+1)$. Thus the first $\leqslant$ in (3) may be replaced by $=$, and similarly the second. This completes the proof of (ii) and (iii) of the Theorem if $1 \leqslant p \leqslant \infty$.

Finally, we prove (iv) of the Theorem. Recall the following facts about Hausdorff measures [4]. Let h: $[0, a) \to [0, \infty)$ be monotone increasing, right continuous, and positive in $(0, a)$ with $h(0)=0$. Then the h-measure of $E \subset \mathbb{R}$ is

$$h-m(E)=\lim_{\delta \to 0^+}\left(\inf\left\{\sum_{i=1}^{\infty} h(d(B_i)) : E \subset \bigcup_{i=1}^{\infty} B_i, \text{ all } d(B_i) \leqslant \delta\right\}\right)$$

where $d(B_i)$ is the length of the interval B_i. The Hausdorff dimension of E is

$$\inf\{\alpha : h-m(E)=0, h(t)=t^\alpha, \alpha>0\}$$

and the logarithmic dimension of E is

$$\inf\{\gamma : h-m(E)=0, h(t)=(\log 1/t)^{-\gamma}, \gamma>0\}$$

provided the set of such γ is non-empty; if it is empty, E is taken to have logarithmic dimension ∞. If E has finite logarithmic dimension, it has zero Hausdorff dimension [4, Theorem 40].

PART (iv) OF THE THEOREM. (i) *For any* $d \in \mathbb{R}$, E_d *has logarithmic dimension* $\leqslant 2$.

(ii) *For any integer* d, E_d *has logarithmic dimension* 2.

Proof. (i) Let $\rho \in (0, 1)$ and consider $E(\rho, d)$. Let $h(t)=(\log 1/t)^{-2-\varepsilon}$, some $\varepsilon>0$. For any positive integer k satisfying $2\rho^k<1$,

$$E(\rho, d) \subset \bigcup_{n=k}^{\infty} \bigcup_{j=0}^{n}\left(\frac{j-d}{n}-\rho^n, \frac{j-d}{n}+\rho^n\right).$$

Hence $E(\rho, d)$ has a cover by intervals B_i s.t. all $d(B_i) \leqslant 2\rho^k$ and s.t.

$$\sum_{i=1}^{\infty} h(d(B_i))=\sum_{n=k}^{\infty}(n+1)\, h(2\rho^n)$$

$$\leqslant 2 \sum_{n=k}^{\infty} nh(2^{n/k}\rho^n)=2\,|\log \rho'|^{-2-\varepsilon} \sum_{n=k}^{\infty} n^{-1-\varepsilon}$$

where $\rho'=2^{1/k}\rho$. Since k is arbitrary, we deduce $h-m(E(\rho, d))=0$ for all $\rho \in (0, 1)$. As we can write $E_d=\bigcup_{n=1}^{\infty} E(1-1/n, d)$, we have $h-m(E_d)=0$. As $\varepsilon>0$ was arbitrary, it follows that E_d has logarithmic dimension $\leqslant 2$.

(ii) We apply Satz 4 in [3, p. 510] with $s=1$. In the notation of this paper, the result is

THEOREM 4. *Let* $w(x)$ *be positive and continuous for* $x \geqslant 1$, *and* $w(x)\, x^2$ *be monotone decreasing for* $x \geqslant 1$. *Further, let* $h(x)$ *be positive, continuous* *and increasing for* $x>0$ *with* $h(0+)=0$, *and let* $h(x)/x$ *be monotone for* $x>0$ *and* $h(2w(x))\,x^2$ *be monotone for* $x\geqslant 1$. *Let*

$$\int_1^\infty h(2w(x))\,x\,dx=\infty. \tag{4}$$

Then $h-m(M(w))=\infty$, *where* $M(w)=\{x\in[0,1)$: *for infinitely many* n, *there exists* $j\leqslant n$ *satisfying* $|x-j/n|<w(n)\}$.

As Jarnik [3, p. 506, footnote 4] remarks, we need assume only that the monotonicity conditions on $w(x)$ hold for large x, since $M(w)$ is independent of the behaviour of $w(x)$ for small or moderate x. Similarly as $h-m$ depends only on the behaviour of $h(t)$ as $t\to 0+$, the monotonicity conditions on $h(t)$ need hold only for small t. Let $\rho\in(0,1)$ and define $w(x)=\rho^x$, $x\in[1,\infty)$ and $h(t)=(\log 1/t)^{-2}$, $t\in(0,1)$. We see $w(x)$ has the requisite monotonicity properties for large x, $h(t)$ has the requisite monotonicity properties for small t, and that (4) holds. By the above results, $h-m(E(\rho,0))=h-m(M(w))=\infty$ and hence $h-m(E_0)=\infty$. Hence E_0 and so E_d for integral d, has logarithmic dimension 2. ∎

The proofs in [3] can probably be modified to show that E_d has logarithmic dimension 2 for any real d.

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